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Variationally trivial Lagrangians and locally variational differential equations of arbitrary order

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Abstract: We continue our investigation of the Lagrangian formalism on jet bundle extensions using Fock space methods. We are able to provide the most general form of a variationally trivial Lagrangian of arbitrary order and we also give a generic expression for the most general locally variational differential equation. As anticipated in the literature, these expressions involve some special combinations of the highest order derivatives, called hyper-Jacobians.

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1. Introduction

In this paper we continue our study [9] of modern Lagrangian theory in the language of jet bundle extensions. We continue to use the finite jet bundle extensions formalism [14–16] combined with Fock space methods. Indeed, various tensors appearing in the proofs have symmetry or antisymmetry properties that make them elements in some Fock space. Then most of the hard combinatorial relations can be written with the help of the creation and annihilation operators and solved using elementary properties of these Fock space operators. In [9] we have used this idea to lay the foundations of the whole theory, i.e., we have given new proofs for the structure formula of the contact and respectively the strong contact forms, for the existence of the Euler–Lagrange and of the Helmholtz–Sonin forms and for the exactness of the variational sequence.

In this paper we continue the analysis with the same methods and give the most general form of a variationally trivial Lagrangian of arbitrary order (in Section 3) and a generic form of a locally variational differential equation (in Sections 4 and 5). In both cases the dependence on the highest order derivatives is through some polynomial expressions, called hyper-Jacobians [4,11], but in the first case we are also able to explicitate the form of the coefficients of these coefficients of these polynomial expressions. The results of Section 3 generalize results from the previous references (see for instance [8]) but some of the formulæ seem to be new in the

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literature. In the second case we reobtain with our method a result from [1]. The paper starts with an Introduction containing the main results from [9] to be used here and finishes with an Appendix about some properties of the Hodge dualisation.

2. Jet bundles and the Lagrangian formalism

This section contains standard material on jet bundle extensions from [9].

2.1. Basic facts about jet bundle extensions

The kinematical structure of a classical field theory is based on fibre bundle structures. Let $\pi : Y \to X$ be a fibre bundle, where X and Y are differentiable manifolds of dimensions $\dim(X) = n$, $\dim(Y) = m + n$ and π is the canonical projection of the fibration. The *adapted charts* to the fibre bundle structure are couples (V, ψ) where V is an open subset of Y and $\psi : V \to \mathbb{R}^n \times \mathbb{R}^m$ is of the form $\psi = (x^i, y^\sigma)$ $(i = 1, \ldots, n; \sigma = 1, \ldots, m)$ such that the canonical projection is locally given by the projection on the first entry. The elements of the *r-jet bundle extension* $J^r_n Y \to X$ $(r \in \mathbb{N})$ are the equivalence classes of sections $j^r_x \gamma$ of the fibre bundle having the same partial derivatives up to order r in the given chart. This new manifold is also a fibre bundle over X, with charts of the form (V^r, ψ^r), where $V^r = (\pi^{r,0})^{-1}(V)$ and

$$\psi = (x^i, y^\sigma, y^\sigma_j, \ldots, y^\sigma_{j_1,\ldots,j_k}, \ldots, y^\sigma_{j_1,\ldots,j_r}), \quad j_1 \leqslant j_2 \leqslant \cdots \leqslant j_k, \ k = 1, \ldots, r;$$

here $y^\sigma_{j_1,\ldots,j_k} \equiv y^\sigma_{j_1,\ldots,j_k}(j^r_x \gamma)$ are the values of the partial derivatives in a given (then in any) local system of coordinates on Y of a given section. We call (V^r, ψ^r) the *associated chart* of (V, ψ). The expressions $y^\sigma_{j_1,\ldots,j_k}(j^r_x \gamma)$ are defined for *all* indices $j_1, \ldots, j_k = 1, \ldots, n$, and the restrictions $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_k$ are in order to avoid overcounting and are a result of the obvious symmetry property

$$y^\sigma_{j_{P(1)},\ldots,j_{P(k)}}(j^r_x \gamma) = y^\sigma_{j_1,\ldots,j_k}(j^r_x \gamma), \tag{2.1}$$

for any permutation $P \in \mathcal{P}_k$, $k = 2, \ldots, r$. To be able to use the summation convention over the dummy indices we consider $y^\sigma_{j_1,\ldots,j_k}$ for *all* values of the indices $j_1, \ldots, j_k \in \{1, \ldots, n\}$ as smooth functions on the chart V^r defined in terms of the independent variables $y^\sigma_{j_1,\ldots,j_k}$, $j_1 \leqslant j_2 \leqslant \ldots \leqslant j_k, k = 1, 2, \ldots, r$, according to the formula (2.1). We make a similar convention for the partial derivatives $\partial/\partial y^\sigma_{j_1,\ldots,j_k}$.

Then we define the vector fields [2]

$$\partial^{j_1,\ldots,j_k}_\sigma \equiv \frac{r_1! \ldots r_n!}{k!} \frac{\partial}{\partial y^\sigma_{j_1,\ldots,j_k}}, \qquad k = 1, \ldots, r \tag{2.2}$$

on the chart V^r for all values of the indices $j_1, \ldots, j_k \in \{1, \ldots, n\}$. Here $r_l, l = 1, \ldots, n$, is the number of times the index l enters into the set $\{j_1, \ldots, j_k\}$. Then we have, for any smooth function f on the chart V^r,

$$df = \frac{\partial f}{\partial x^i} dx^i + \sum_{k=0}^{r} (\partial^{j_1,\ldots,j_k}_\sigma f)\, dy^\sigma_{j_1,\ldots,j_k} = \frac{\partial f}{\partial x^i} dx^i + \sum_{|J| \leqslant r} (\partial^J_\sigma f)\, dy^\sigma_J. \tag{2.3}$$

In the last formula we have introduced the multi-index notations in an obvious way. We now define the *formal derivatives*

$$d_i^r \equiv \frac{\partial}{\partial x^i} + \sum_{k=0}^{r-1} y^\sigma_{i,j_1,\ldots,j_k} \partial_\sigma^{j_1,\ldots,j_k}. \tag{2.4}$$

Sometimes, when no danger of confusion is possible, we denote more simply $d_i = d_i^r$.

If γ is a section of the fibre bundle Y then we denote by $j^r\gamma : V \to J^rY$ its *r-extension* which is a section of the r-jet bundle extension.

We denote the forms of degree q on J^rY by Ω^r_q. A form $\rho \in \Omega^r_q$ is called π^r*-horizontal* (or *basic*) if it has the local expression

$$\rho = B_{i_1,\ldots,i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q} \tag{2.5}$$

with $B_{i_1,\ldots,i_q}$ smooth skew-symmetric functions on V^r. We denote the set of basic forms of degree q by $\Omega^r_{q,X}$. The elements of $\lambda \in \Omega^r_{n,X}$ are called *Lagrange forms*. They have the local expression

$$\lambda = L\theta_0 \tag{2.6}$$

where L is a smooth function on V^r and

$$\theta_0 \equiv dx^1 \wedge \cdots \wedge dx^n. \tag{2.7}$$

By a *contact form* we mean any form $\rho \in \Omega^r_q$ satisfying

$$\left(j^r\gamma\right)^* \rho = 0 \tag{2.8}$$

for any section γ. We denote by $\Omega^r_{q(c)}$ the set of contact forms of degree $q \leqslant n$. When considering the contact forms on an open set $V \subset Y$ only, then we emphasize this by writing $\Omega^r_{q(c)}(V)$. One immediately notes that $\Omega^r_{0(c)} = 0$ and that for $q > n$ any q-form is contact. It is also elementary to see that the set of all contact forms is an ideal, denoted by $\mathcal{C}(\Omega^r)$, with respect to the operation $\wedge$. For any chart (V, ψ) on Y, by elementary computations one finds that every element of the set $\Omega^r_{1(c)}(V)$ is a linear combination of the expressions

$$\omega^\sigma_{j_1,\ldots,j_k} \equiv dy^\sigma_{j_1,\ldots,j_k} - y^\sigma_{i,j_1,\ldots,j_k} dx^i, \quad k = 0, \ldots, r-1. \tag{2.9}$$

If $\rho \in \Omega^r_q$, $q > 1$, then we denote its *contact component of order k* ($k = 0, \ldots, q$) by $p_k\rho$. The operator $h \equiv p_0$ is called the *horizontalization*; one can prove that the form ρ is contact *iff* $h\rho = 0$. One can provide some very explicit expressions for the components $p_k\rho$ for any form ρ (see [9]).

Let us notice that the definition of the contact forms is trivially satisfied if the degree of the form is $q \geqslant n+1$. So, it is natural to try a generalization of the contact forms in this case. It seems plausible to use, instead of the horizontalization operator h, some other projection p_k from those introduced above. The proper definition is the following one. Let $q = n+1, \ldots, N \equiv \dim(J^rY) = m\binom{n+r}{n}$ and let $\rho \in \Omega^r_q$. One says that ρ is a *strongly contact form iff* its contact component of order $q - n$ vanishes, i.e.,

$$p_{q-n}\rho = 0. \tag{2.10}$$

For uniformity of notation, we also denote these forms by $\Omega^r_{q(c)}$.

2.2. Fock space methods

The rôle of the Fock space methods emerges from a canonical expression for any form $\rho \in \Omega^r_q$. We start from the fact that the forms dx^i, ω^σ_J ($|J| \leqslant r-1$) and dy^σ_I ($|I| = r$) are a basis in the linear space of 1-forms. The form ρ is a polynomial of degree q in these forms (with respect to the product $\wedge$). We separate all the terms containing at least one factor ω^σ_J and get a decomposition

$$\rho = \rho_0 + \rho' \tag{2.11}$$

where ρ_0 has the structure

$$\rho_0 = \sum_{|J| \leqslant r-1} \omega^\sigma_J \wedge \Phi^J_\sigma \tag{2.12}$$

and ρ' is a polynomial of degree q in dx^i and dy^σ_I ($|I| = r$) only . It is clear that one can write

$$\rho' = \sum_{s=0}^{q} \frac{1}{s!\,(q-s)!} \sum_{|I_1|,\dots,|I_s|=r} A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_q}\, dy^{\sigma_1}_{I_1} \wedge \cdots \wedge dy^{\sigma_s}_{I_s} \wedge dx^{i_{s+1}} \wedge \cdots \wedge dx^{i_q}, \tag{2.13}$$

where $A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_q}$ are smooth functions on V^r and can be assumed to satisfy the following symmetry property

$$A^{I_{P(1)},\dots,I_{P(s)}}_{\sigma_{P(1)},\dots,\sigma_{P(s)},i_{Q(s+1)},\dots,i_{Q(q)}} = (-1)^{|P|+|Q|} A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_q}, \tag{2.14}$$

where P and Q are permutations and $|P|$, $|Q|$ are their signatures.

Let us define the Hilbert spaces of Fock type

$$\mathcal{H}_s \equiv \mathcal{F}^{(-)}(\mathbb{R}^n) \otimes \underbrace{\mathcal{F}^{(+)}(\mathbb{R}^n) \otimes \cdots \otimes \mathcal{F}^{(+)}(\mathbb{R}^n)}_{s}$$

where $\mathcal{F}^{(\pm)}(\mathbb{R}^n)$ are the symmetric (corresp. +) and the antisymmetric (corresp. −) Fock spaces. Then we have the well-known decomposition in subspaces with fixed number of "bosons" and "fermions": $\mathcal{H}_s = \bigoplus \mathcal{H}_{k,l_1,\dots,l_s}$ with the convention: $\mathcal{H}_{k,l_1,\dots,l_s} \equiv 0$ if any one of the indices $k, l_1, \dots, l_q$ is negative or if $k > n$. Then we can consider $A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_q}$ as the components of a tensor

$$A_{\sigma_1,\dots,\sigma_s} \in \mathcal{H}_{q-s,\underbrace{r,\dots,r}_{s}}.$$

Let us denote the creation and the annihilation fermionic operators a^{*i}, a_i ($i = 1, \dots, n$) and the corresponding creation and annihilation bosonic operators $b^*_{(\alpha)i}, b^i_{(\alpha)}$ ($\alpha = 1, \dots, s$; $i = 1, \dots, n$). One introduces the operators

$$B_\alpha : \mathcal{H}_{k,l_1,\dots,l_s} \to \mathcal{H}_{k+1,l_1,\dots,l_{\alpha-1},l_\alpha+1,l_{\alpha+1},\dots,l_s} \quad (\alpha = 1, \dots, s)$$

according to the formula

$$B_\alpha \equiv b^*_{(\alpha)i} a^{*i}, \qquad \alpha = 1, \ldots, s. \tag{2.15}$$

It is convenient to rescale these operators (compare with [9]) so that we have for instance

$$(B_1 A)^{\{j_0,\ldots,j_k\},I_2,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s,i_0,\ldots,i_l} = (-1)^{s-1} \mathcal{S}^+_{j_0,\ldots,j_k} \mathcal{S}^-_{i_0,\ldots,i_l} A^{\{j_1,\ldots,j_k\},I_2,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s,i_1,\ldots,i_l} \delta^{j_0}_{i_0} \tag{2.16}$$

and similarly for $B_2, \ldots, B_s$; here $\mathcal{S}^\pm_J$ is the symmetrization (resp. antisymmetrization) operator in the indices J. The central property of these operators is (see [9, Lemma 3.6]):

Lemma 2.1. *Let $X \in \mathcal{H}_{k,r_1,\ldots,r_s}$. Then X satisfies the equation*

$$B_1 \cdots B_s X = 0 \tag{2.17}$$

iff it is of the form

$$X = \sum_{\alpha=1}^{s} B_\alpha X_\alpha \tag{2.18}$$

for some $X_\alpha \in \mathcal{H}_{k-1,r_1,\ldots,r_{\alpha-1},r_\alpha-1,r_{\alpha+1},\ldots,r_s}$.

We note that the proof of this result is not affected by the rescaling performed above.

This key result was used in [9] to obtain a structure formula for an arbitrary contact form generalizing the expression (2.9) for contact forms of degree 1. Explicitly, let (V, ψ) be an adapted chart on the fibre bundle Y and let $\rho \in \Omega^r_q(Y)$, $q = 2, \ldots, n$. Then ρ is contact *iff* in the associated chart is a sum of terms each of them having at least one factor which is a contact form of degree 1, i.e., ω^σ_J, $|J| \leqslant r-1$, or an exterior differential of the form $d\omega^\sigma_I$, $|I| = r$. Applying the same methods one obtains the structure theorem for strongly contact forms. Let $n+1 \leqslant q \leqslant N$ and $\rho \in \Omega^r_q$. Let (V, ψ) be a chart on Y Then ρ is a strongly contact form *iff* in the associated chart it is a sum of terms each of them having at least $q-n+1$ factors of the type ω^σ_J, $|J| \leqslant r-1$, or $d\omega^\sigma_I$, $|I| = r$.

An element $T \in \Omega^s_{n+1,X}$ is called a *differential equation* (or a *source form* in the terminology of [1]) if it has the following local expression in every associated chart:

$$T = T_\sigma \omega^\sigma \wedge \theta_0. \tag{2.19}$$

When one considers the so-called variationally trivial Lagrangians the following subset of the space of basic forms emerges

$$\mathcal{J}^r_q \equiv \{\rho \in \Omega^r_{q,X} \mid \exists \nu \in \Omega^{r-1}_q \text{ such that } \rho = h\nu\}. \tag{2.20}$$

One notes that if $\rho_i \in \mathcal{J}^r_{q_i}$, $i = 1, 2$, then $\rho_1 \wedge \rho_2 \in \mathcal{J}^r_{q_1+q_2}$; also the operator $D : \mathcal{J}^r_q \to \mathcal{J}^r_{q+1}$ given by

$$Dh\nu \equiv hd\nu \tag{2.21}$$

is well defined [2]. The operator D is called the *total exterior derivative*.

2.3. Euler–Lagrange and Helmholtz–Sonin forms

Let us take $\lambda \in \Omega^r_{n,X}$ with the local expression (2.6). Then one defines, on any chart (V^s, ψ^s), $s \geqslant 2r$, the so-called *Lie–Euler expressions* [1] according to

$$E^I_\sigma(L) \equiv \sum_{|J| \leqslant r-|I|} (-1)^{|J|} \binom{|I|+|J|}{|I|} d_J \partial^{IJ}_\sigma L. \tag{2.22}$$

In particular, for $I = \emptyset$ we have the *Euler–Lagrange expressions*

$$E_\sigma(L) \equiv \sum_{|J| \leqslant r} (-1)^{|J|} d_J \partial^J_\sigma L. \tag{2.23}$$

Then there exists a globally defined $(n+1)$-form, denoted by $E(\lambda)$, such that in V^s

$$E(\lambda) = E_\sigma(L)\, \omega^\sigma \wedge \theta_0. \tag{2.24}$$

One calls this form the *Euler–Lagrange form* associated to λ and notes that it is a differential equation (see formula (2.19)). In general, a differential equation $T \in \Omega^s_{n+1,X}$ is called (*locally*) *variational* or *of Euler–Lagrange type iff* there exists a (local) Lagrange form $\lambda \in \Omega^r_{n,X}$ such that

$$T = E(\lambda), \qquad \text{up to a pull-back.} \tag{2.25}$$

One notices that in this case the general form of a differential equation coincides with the well-known form of the Euler–Lagrange equations.

We also remark the following property of the Euler–Lagrange form. Let A^I, $|I| = l \geqslant 1$, be some smooth functions on V^r and $f \equiv d_I A^I$. Then we have on V^s, $s \geqslant 2(r+l)$,

$$E^J_\sigma(f) = 0, \qquad |J| \leqslant |I| - 1. \tag{2.26}$$

A similar construction provides the so-called Helmholtz–Sonin form. Let $T \in \Omega^s_{n+1,X}$ be a differential equation with the local form given by (2.19). We define the following expressions in any chart V^t, $t \geqslant 2s$:

$$H^J_{\sigma\nu} \equiv \partial^J_\nu T_\sigma - (-1)^{|J|} E^J_\sigma(T_\nu), \qquad |J| \leqslant s. \tag{2.27}$$

Then there exists a globally defined $(n+2)$-form, denoted by $H(T)$, such that on V^t

$$H(T) = \sum_{|J| \leqslant s} H^J_{\sigma\nu}\, \omega^\nu_J \wedge \omega^\sigma \wedge \theta_0. \tag{2.28}$$

2.4. The exact variational sequence

The basic construction goes along the following lines. If $\pi : Y \to X$ is a fibre bundle and $U, V \subset Y$ are charts such that $U \subset V$, we denote by $i_{U,V} : U^r \to V^r$ the canonical inclusion. Then the collection $\{\Omega^r_q(V)\}$ $(q \geqslant 0)$, $\{i^*_{U,V}\}$ is a presheaf denoted by Ω^r_q. One also needs the subspaces $\theta^r_q \in \Omega^r_{q(c)}$ defined by

$$\theta^r_1 \equiv \Omega^r_{1(c)}, \qquad \theta^r_q \equiv d\Omega^r_{q-1(c)} + \Omega^r_{q(c)}, \qquad q = 2, \ldots, N = \dim(J^r Y). \tag{2.29}$$

Next, one introduces the so-called contact homotopy operator. Let $U \subset \mathbb{R}^n$ (resp. $V \subset \mathbb{R}^m$) be an open set (resp. a ball centred in $0 \in \mathbb{R}^m$) and $W = U \times V$. One considers the operator χ_t as a map $\chi_t^r : [0, 1] \times J^r W \to J^r W$ given by

$$\chi_r(t, (x^i, y^\sigma, y^\sigma_j, \ldots, y^\sigma_{j_1,\ldots,j_s})) = (x^i, ty^\sigma, ty^\sigma_j, \ldots, ty^\sigma_{j_1,\ldots,j_s}). \tag{2.30}$$

Then for any $\rho \in \Omega^r_q(W)$ we have the *unique* decomposition

$$(\chi_t^r)^* \rho = dt \wedge \rho_0(t) + \rho_1(t) \tag{2.31}$$

where $\rho_0(t)$ (resp. $\rho_1(t)$) are $q - 1$ (resp. q) forms which do not contain the differential dt. Then the *contact homotopy operator* is by definition the map $A : \Omega^r_q(V) \to \Omega^r_{q-1}(V)$ given by

$$A\rho \equiv \int_0^1 \rho_0(t). \tag{2.32}$$

This definition is justified by the following result resembling the key idea of the Poincaré lemma. Let $\rho \in \Omega^r W$ be arbitrary. The following formula is true:

$$\rho = Ad\rho + dA\rho + \rho_1(0). \tag{2.33}$$

The contact homotopy is essential for the proof of the central result of [15], which is

Theorem 2.2. *Let $E_q : \Omega^r_q/\theta^r_q \to \Omega^r_{q+1}/\theta^r_{q+1}$ be given by*

$$E_q([\rho]) \equiv [d\rho] \tag{2.34}$$

where $[\rho]$ is the class of ρ modulo θ^r_q. Then the quotient sequence

$$0 \to \mathbb{R} \to \Omega^r_0 \xrightarrow{E_0} \Omega^r_1/\theta^r_1 \xrightarrow{E_1} \cdots \xrightarrow{E_{P-1}} \Omega^r_P/\theta^r_P \xrightarrow{E_P} \Omega^r_{P+1} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^r_N \to 0 \tag{2.35}$$

is an acyclic resolution of the constant sheaf $\mathbb{R}$. In particular it is exact.

One calls (2.35) the *variational sequence of order r over Y* and denotes for simplicity: $\mathcal{V}^r_q \equiv \Omega^r_q/\theta^r_q$. Some special classes have distinct names. So, if $\lambda \in \Omega^r_n$, then the class $[\lambda] \in \Omega^r_{n+1}/\theta^r_{n+1}$ is called the *Euler–Lagrange class* of λ. If $T \in \Omega^r_{n+1}$ then $[T] \in \Omega^r_{n+2}/\theta^r_{n+2}$ is called the *Helmholtz–Sonin class* of T.

Let us note in the end of this subsection that for any $q = n + 1, \ldots, N$, $s > r$ there exists a canonical isomorphism

$$i_{s,r} : \Omega^r_q/\theta^r_q \to \operatorname{Im}\left(\tau^s_q \circ (\pi^{s,r+1})^* \circ p_{q-n}\right),$$

where $\tau^s_q : \Omega^s_q \to \Omega^s_q/\theta^s_q$ is the canonical projection. The explicit expression is

$$i_{s,r}([\rho]) = \tau^s_q \circ (\pi^{s,r+1})^* \circ p_{q-n}(\rho) \tag{2.36}$$

and this definition is consistent [15]. One can give characterizations of $\mathcal{V}^r_q$, $q = n, n+1$ using Euler–Lagrange and Helmholtz–Sonin forms. The first result is

Theorem 2.3. *If $\lambda \in \Omega^r_{n,X}$ is any Lagrange form, then for any $s \geqslant 2r$ we have*

$$i_{s,r}(E_n([\lambda])) = [E(\lambda)] \tag{2.37}$$

where $E(\lambda)$ is the Euler–Lagrange form associated to λ (see (2.23) *and* (2.24)*).*

Similarly we have:

Theorem 2.4. *Let $T \in \Omega^s_{n+1,X}$ be a differential equation. The for any $t \geqslant 2s$ we have*

$$i_{t,s}(E_{n+1}([T])) = \left[\tfrac{1}{2}H(T)\right] \tag{2.38}$$

where $H(T)$ is the Helmholtz–Sonin form associated to T.

3. Variationally trivial Lagrangians

A *variationally trivial Lagrange form of order r* is any $\lambda \in \Omega^r_{n,X}$ such that $E(\lambda) = 0$. In other words, the corresponding Euler–Lagrange expressions are identically zero, or the Euler–Lagrange equations are identities for any section γ. Here we give the general form of such a Lagrange form. We follow, basically, the line of argument from [16]; as before, we simplify considerably and complete the proofs from this reference using the techniques developed in [9]. The main reward is the possibility to obtain a very explicit expression for a locally trivial Lagrangian.

First, we recall that $\rho \in \Omega^{r+1}_q$ is $\pi^{r+1,r}$*-projectable iff* there exists $\rho' \in \Omega^r_q$ such that

$$\rho = (\pi^{r+1,r})^*\rho'. \tag{3.1}$$

One can easily see that locally this amounts to the condition that if ρ is written as a polynomial in dx^i and dy^σ_J ($|J| \leqslant r+1$), then the differentials dy^σ_J ($|J| = r+1$) must be absent and moreover, the coefficient functions must not depend on y^σ_J ($|J| = r+1$).

We start with the following result [15]:

Proposition 3.1. *Let $\eta \in \Omega^r_q$ ($q = 1, \ldots, n-1$) such that $hd\eta$ is a $\pi^{r+1,r}$-projectable $(q+1)$-form. Then one can write η as*

$$\eta = \nu + d\phi + \psi \tag{3.2}$$

where $\nu \in \Omega^r_{q,X}$ is a basic q-form and $\psi \in \Omega^r_{q(c)}$ is a contact q-form.

Proof. Let (V, ψ) be a chart on Y. Then in the associated chart (V^r, ψ^r) one can write η in the standard form

$$\eta = \eta_0 + \eta_1 \tag{3.3}$$

where in η_0 we collect all terms containing at least one factor ω^σ_J ($|J| \leqslant r-1$) and η_1 is a polynomial in dx^i and dy^σ_J ($|J| = r+1$) only,

$$\eta_1 = \sum_{s=0}^{q} \sum_{|I_1|,\ldots,|I_s|=r} A^{I_1,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s,i_{s+1},\ldots,i_q}\, dy^{\sigma_1}_{I_1} \wedge \cdots \wedge dy^{\sigma_s}_{I_s} \wedge dx^{i_{s+1}} \wedge \cdots \wedge dx^{i_q}, \tag{3.4}$$

where the coefficients $A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_q}$ have antisymmetry properties of type (2.14).

In particular η_0 is a contact form so we have $hd\eta = hd\eta_1$, i.e., the form $hd\eta_1$ is, by hypothesis, $\pi^{r+1,r}$-projectable. One first computes in general

$$d\eta_1 = \sum_{s=0}^{q} \sum_{|I_1|,\dots,|I_s|=r} \sum_{|J|\leqslant r-1} \left(\partial^J_\nu A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_q}\right) \omega^\nu_J \wedge dy^{\sigma_1}_{I_1} \wedge \cdots \wedge dy^{\sigma_s}_{I_s} \wedge dx^{i_{s+1}} \wedge \cdots \wedge dx^{i_q}$$
$$+ \sum_{s=0}^{q+1} \sum_{|I_1|,\dots,|I_s|=r} \tilde{A}^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_{q+1}} dy^{\sigma_1}_{I_1} \wedge \cdots \wedge dy^{\sigma_s}_{I_s} \wedge dx^{i_{s+1}} \wedge \cdots \wedge dx^{i_{q+1}},$$

where

$$\tilde{A}^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_{q+1}} \equiv \frac{1}{s} \sum_{k=1}^{s} (-1)^{k-1} \partial^{I_k}_{\sigma_k} A^{I_1,\dots,\hat{I}_k,\dots,I_s}_{\sigma_1,\dots,\hat{\sigma}_k,\dots,\sigma_s,i_{s+1},\dots,i_{q+1}}$$
$$+ \frac{1}{q+1-s} \sum_{k=s+1}^{q+1} (-1)^{k-1} d_{i_k} A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,\hat{i}_k,\dots,i_{q+1}},$$

or, equivalently

$$\begin{aligned} \tilde{A}^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_{q+1}} &\equiv \mathcal{S}^-_{(I_1,\sigma_1),\dots,(I_s,\sigma_s)} \partial^{I_1}_{\sigma_1} A^{I_2,\dots,I_s}_{\sigma_2,\dots,\sigma_s,i_{s+1},\dots,i_{q+1}} \\ &+ (-1)^s \mathcal{S}^-_{i_{s+1},\dots,i_{q+1}} d_{i_{s+1}} A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+2},\dots,i_{q+1}} \end{aligned} \tag{3.5}$$

where $\mathcal{S}^-_{(I_1,\sigma_1),\dots,(I_s,\sigma_s)}$ is the antisymmetrization projector in the corresponding couples of indices.

Then one gets

$$hd\eta = hd\eta_1 = \sum_{s=0}^{q+1} \sum_{|I_1|,\dots,|I_s|=r} \tilde{A}^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_{q+1}} y^{\sigma_1}_{I_1 i_1} \cdots y^{\sigma_s}_{I_s i_s} dx^{i_1} \wedge \cdots \wedge dx^{i_{q+1}}$$

which is $\pi^{r+1,r}$-projectable *iff* the coefficient functions do not depend on y^σ_I $(|I| = r+1)$.

(ii) By repeatedly applying derivative operators, one obtains, as in [9], that the condition of $\pi^{r+1,r}$-projectability is equivalent to

$$\mathcal{S}^-_{i_1,\dots,i_{q+1}} \mathcal{S}^+_{I_1 p_1} \cdots \mathcal{S}^+_{I_s p_s} \tilde{A}^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_{q+1}} \delta^{p_1}_{i_1} \cdots \delta^{p_s}_{i_s} = 0, \quad s = 1,\dots,q+1$$

or, using a familiar argument,

$$B_1 \cdots B_s \tilde{A}_{\sigma_1,\dots,\sigma_s} = 0, \quad s = 1,\dots,q+1. \tag{3.6}$$

It is possible to apply Lemma 2.1 and one obtains the generic expression

$$\tilde{A}_{\sigma_1,\dots,\sigma_s} = \sum_{\alpha=1}^{s} B_\alpha \tilde{A}^\alpha_{\sigma_1,\dots,\sigma_s}.$$

In index notation, this means that the function $\tilde{A}^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_{q+1}}$ is a sum of terms containing at least one factor of the type δ^j_i where j belongs to one of the multi-indices I_p, $p = 1,\dots,s$, and $i \in \{i_{s+1},\dots,i_{q+1}\}$.

If we insert back into the expression of $d\eta_1$, we get after minor prelucrations that

$$d\eta_1 = \nu_1 + d\phi + \sum_{|J|\leqslant r-1} \omega_J^\sigma \wedge \Phi_\sigma^J \tag{3.7}$$

with $\nu_1 \in \Omega^r_{q,X}$ a basic q-form, $\phi \in \Omega^r_{q(c)}$ and Φ_σ^J a polynomial of degree q in dx^i and dy_I^σ, $|I| = r$.

This relation implies that

$$d\nu_1 + \sum_{|J|\leqslant r-1} \left(d\omega_J^\sigma \wedge \Phi_\sigma^J - \omega_J^\sigma \wedge d\Phi_\sigma^J \right) = 0. \tag{3.8}$$

One applies the operator p_1 to this equality; using [9, Lemma 4.1] a new relation is obtained:

$$p_1 d\nu_1 + \sum_{|J|\leqslant r-1} \left(d\omega_J^\sigma \wedge h\Phi_\sigma^J - \omega_J^\sigma \wedge hd\Phi_\sigma^J \right) = 0. \tag{3.9}$$

One must consider now the generic forms for ν_1 and Φ_σ^J, namely

$$\nu_1 = A_{i_1,\ldots,i_{q+1}}\, dx^{i_1} \wedge \cdots \wedge dx^{i_{q+1}} \tag{3.10}$$

and

$$\Phi_\sigma^J = \sum_{s=0}^{q} \sum_{|I_1|,\ldots,|I_s|=r} \Phi^{J,I_1,\ldots,I_s}_{\sigma,\nu_1,\ldots,\nu_s,i_{s+1},\ldots,i_q}\, dy_{I_1}^{\nu_1} \wedge \cdots \wedge dy_{I_s}^{\nu_s} \wedge dx^{i_{s+1}} \wedge \cdots \wedge dx^{i_q}; \tag{3.11}$$

here we can assume a (partial) symmetry property of the type (2.14)

$$\Phi^{J,I_{P(1)},\ldots,I_{P(s)}}_{\sigma,\nu_{P(1)},\ldots,\nu_{P(s)},i_{Q(s+1)},\ldots,i_{Q(q)}} = (-1)^{|P|+|Q|}\, \Phi^{J,I_1,\ldots,I_s}_{\sigma,\nu_1,\ldots,\nu_s,i_{s+1},\ldots,i_q} \tag{3.12}$$

and we make the convention that

$$\Phi^{J,I_1,\ldots,I_s}_{\sigma,\nu_1,\ldots,\nu_s,i_{s+1},\ldots,i_q} = 0, \qquad \forall J \text{ such that } |J| \geqslant r. \tag{3.13}$$

These expression must be plugged into the equation (3.9). One finds from this equation that

$$\begin{aligned} &\mathcal{S}^-_{i_1,\ldots,i_{q+1}} \mathcal{S}^+_{I_1 p_1} \cdots \mathcal{S}^+_{I_s p_s} \left[\tilde{\Phi}'^{\{j_1,\ldots,j_k\},I_1,\ldots,I_s}_{\sigma,\nu_1,\ldots,\nu_s,i_{s+1},\ldots,i_{q+1}} + (-1)^s \mathcal{S}_{j_1,\ldots,j_k}\, \delta^{j_1}_{i_{s+1}}\, \Phi^{\{j_2,\ldots,j_k\},I_1,\ldots,I_s}_{\sigma,\nu_1,\ldots,\nu_s,i_{s+2},\ldots,i_{q+1}} \right] \\ &\quad \times \delta^{p_1}_{i_1} \cdots \delta^{p_s}_{i_s} = 0, \qquad s = 1,\ldots,q+1,\ k = 0,\ldots,r. \end{aligned} \tag{3.14}$$

Here we have defined, in analogy with (3.5),

$$\begin{aligned} \tilde{A}'^{I_0,\ldots,I_s}_{\sigma_0,\ldots,\sigma_s,i_{s+1},\ldots,i_{q+1}} &\equiv \mathcal{S}^-_{(I_1,\sigma_1),\ldots,(I_s\sigma_s)} \partial^{I_1}_{\sigma_1} A^{I_0,I_2,\ldots,I_s}_{\sigma_0,\sigma_2,\ldots,\sigma_s,i_{s+1},\ldots,i_{q+1}} \\ &\quad + (-1)^s \mathcal{S}^-_{i_{s+1},\ldots,i_{q+1}} d_{i_{s+1}} A^{I_0,I_1,\ldots,I_s}_{\sigma_0,\ldots,\sigma_s,i_{s+2},\ldots,i_{q+1}}. \end{aligned} \tag{3.15}$$

To simplify the analysis of the relation (3.14), one observes that it is possible to define a map

$$\Delta' : \bigoplus_{s=0}^{q} \mathcal{H}_{q-s,k,\underbrace{r,\ldots,r}_{s}} \to \bigoplus_{s=0}^{q+1} \mathcal{H}_{q+1-s,k,\underbrace{r,\ldots,r}_{s}}$$

according to

$$(\Delta'\Phi)^{J,I_1,\ldots,I_s}_{\sigma,\nu_1,\ldots,\nu_s,i_{s+1},\ldots,i_{q+1}} = \tilde{\Phi}'^{J,I_1,\ldots,I_s}_{\sigma,\nu_1,\ldots,\nu_s,i_{s+1},\ldots,i_{q+1}}. \tag{3.16}$$

Then the equation above takes the form

$$B_1 \cdots B_s \left((\Delta\Phi)_s^k + B_0 \Phi_s^{k-1} \right) = 0, \quad s = 1, \ldots, q+1, \ k = 0, \ldots, r \tag{3.17}$$

where

$$\Phi_s^k \in \mathcal{H}_{q-s,k,\underbrace{r,\ldots,r}_{s}}$$

is the tensor of components $\Phi^{J,I_1,\ldots,I_s}_{\sigma,\nu_1,\ldots,\nu_s,i_{s+1},\ldots,i_q}$, $|J| = k$.

We remind the reader that we have made the convention

$$\Phi_s^{-1} = 0, \quad \Phi_s^r = 0. \tag{3.18}$$

To solve the preceding equation one first establishes by direct computation that

$$\{\Delta', B_0\} = 0 \tag{3.19}$$

and

$$(\Delta')^2 = 0. \tag{3.20}$$

Now we can solve the system (3.17) by a procedure similar to the descent procedure applied in the study of gauge theories, for instance, in the BRST analysis of anomalies [18]. We take $k = r$ in (3.17) and, under the convention (3.18), we obtain

$$B_0 \cdots B_s \Phi_s^{r-1} = 0, \quad s = 1, \ldots, q. \tag{3.21}$$

Lemma 2.1 can be applied and it follows that we have a generic expression of the form

$$\Phi_s^{r-1} = \sum_{\alpha=0}^{s} B_\alpha \Phi_s^{r-1,\alpha}, \quad s = 1, \ldots, q.$$

If we substitute this expression into the initial expression (3.11) of Φ_σ^J, $|J| = r-1$, we find that the contributions corresponding to $\alpha = 1, \ldots, s$ are producing contact terms. So, we can redefine the expressions Φ_σ^J so that instead of (3.7) we have

$$d\eta_1 = \nu_1 + d\phi + \sum_{|J| \leqslant r-1} \omega_J^\sigma \wedge \Phi_\sigma^J + \psi \tag{3.22}$$

with ψ some 2-contact form and moreover

$$\Phi_s^{r-1} = B_0 C_s^{r-1}, \quad s = 1, \ldots, q \tag{3.23}$$

for some tensors

$$C_s^{r-1} \in \mathcal{H}_{q-s-1,r-2,\underbrace{r,\ldots,r}_{s}}.$$

The procedure can now be iterated taking $k = r-1$ in (3.17) etc. The result of this descent procedure is the following: one can redefine the tensors Φ_σ^J so that we have (3.22) and

$$\Phi_s^k = B_0 C_s^k + (\Delta C^{k+1})_s, \quad s = 1, \ldots, q; \quad k = 0, \ldots, r-1 \tag{3.24}$$

for some tensors

$$C^k_s \in \mathcal{H}_{q-s-1,k-1,\underbrace{r,\ldots,r}_{s}};$$

by convention

$$C^0_s = 0, \quad C^r_s = 0, \quad s = 1, \ldots, q.$$

In full index notation this means that we have

$$\begin{aligned} \Phi^{\{j_1,\ldots,j_k\},I_1,\ldots,I_s}_{\sigma,\nu_1,\ldots,\nu_s,i_{s+1},\ldots,i_q} &= (-1)^s \mathcal{S}^+_{j_1,\ldots,j_k} \mathcal{S}^-_{i_{s+1},\ldots,i_q} \delta^{j_1}_{i_{s+1}} C^{\{j_2,\ldots,j_k\},I_1,\ldots,I_s}_{\sigma,\nu_1,\ldots,\nu_s,i_{s+2},\ldots,i_q} \\ &+ \tilde{C}'^{\{j_1,\ldots,j_k\},I_1,\ldots,I_s}_{\sigma,\nu_1,\ldots,\nu_s,i_{s+1},\ldots,i_q}, \quad s = 1,\ldots,q;\ k = 0,\ldots,r. \end{aligned} \tag{3.25}$$

Now one can substitute this expression for the tensors Φ into the original expression for the forms (3.11). After some algebra one obtains that

$$\sum_{|J|\leqslant r-1} \omega^\sigma_J \wedge \Phi^J_\sigma = \sum_{|J|\leqslant r-1} \Phi^J_{\sigma,i_1,\ldots,i_q} \omega^\sigma_J \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q} + d\phi_1 \tag{3.26}$$

where $\phi_1 \in \Omega^r_{q(c)}$ is a contact form. It emerges that the relation (3.22) becomes

$$d\eta_1 = \nu_1 + d\phi_1 + \sum_{|J|\leqslant r-1} \Phi^J_{\sigma,i_1,\ldots,i_q} \omega^\sigma_J \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q} + \psi \tag{3.27}$$

where ϕ_1 (resp. ψ) is some contact (resp. 2-contact) form of degree q (resp. $q+1$).

(iii) The last step of our proof consists in using the contact homotopy operator A (see the definition contained in the relations (2.30)–(2.32)). We apply the relation (2.33) to the form $\rho = \eta_1 - \phi_1$

$$\begin{aligned} \eta_1 - \phi_1 = A\Big(\nu_1 + \sum_{|J|\leqslant r-1} \Phi^J_{\sigma,i_1,\ldots,i_q} \omega^\sigma_J \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q} + \psi\Big) \\ + dA(\eta_1 - \phi_1) + \eta_1(0) - \phi_1(0). \end{aligned} \tag{3.28}$$

One proves by direct computation the following facts

- $A\nu_1 = 0$;
- if ψ is 2-contact form, then $A\psi$ is a contact form;
- $A(\sum \cdots)$ and $\eta_1(0)$ are basic forms;
- if ϕ_1 is a contact form, then $\phi_1(0)$ is also a contact form.

Inserting this information in the preceding relation one obtains that the result from the statement of the proposition is true for the form η_1. Taking into account (3.3) we obtain the same result for the form η. □

Let us note that the converse of this statement is not true. In fact the condition of $\pi^{r+1,r}$-projectability imposes additional constraints on the basic form ν which will be analysed in the next lemma. A complete proof of the following result appears in [2]; here we offer an alternative proof which seems to be much more simpler.

Proposition 3.2. *Let $\nu \in \Omega^r_{q,X}$ ($q = 1,\ldots,n-1$). Then the basic form $hd\nu$ is $\pi^{r+1,r}$-projectable iff there exists, for any chart (V,ψ), a form $\tilde{\nu}_V \in \Omega^{r-1}_q$ such that we have the equality $\nu = h\tilde{\nu}_V$ in the associated chart (V^r, ψ^r).*

Proof. According to (2.5), in the associated chart (V^r, ψ^r) we have

$$\nu = A_{i_1,\dots,i_q}\, dx^{i_1} \wedge \cdots \wedge dx^{i_q}.$$

Then one obtains by direct computation that

$$hd\nu = \Big[\left(d_{i_0} A_{i_1,\dots,i_q}\right) + \sum_{|I|=r} \left(\partial^I_\sigma A_{i_1,\dots,i_q}\right) y^\sigma_{Ii_0} \Big] dx^{i_0} \wedge \cdots \wedge dx^{i_q}$$

(here d_j is the formal derivative on V^r).

This form is $\pi^{r+1,r}$-projectable *iff* the square bracket does not depend on y^σ_I $(|I| = r+1)$, i.e.,

$$\mathcal{S}^-_{i_0,\dots,i_q} \mathcal{S}^+_{Ip} \delta^p_{i_0} \left(\partial^I_\sigma A_{i_1,\dots,i_q}\right) = 0. \tag{3.29}$$

It is clear that this relation is very similar to those already analysed. To be able to apply the central result contained in Lemma 2.1 one must make a little trick. From the previous relation we obtain by derivation

$$\mathcal{S}^-_{i_0,\dots,i_q} \mathcal{S}^+_{I_0 p_0} \delta^{p_0}_{i_0} \left(\partial^{I_0}_{\sigma_0} \cdots \partial^{I_q}_{\sigma_q} A_{i_1,\dots,i_q}\right) = 0. \tag{3.30}$$

If we define the tensor

$$A_{\sigma_0,\dots,\sigma_q} \in \mathcal{H}_{q,\underbrace{r,\dots,r}_{q+1}}$$

by

$$A^{I_0,\dots,I_q}_{\sigma_0,\dots,\sigma_q,i_1,\dots,i_q} \equiv \partial^{I_0}_{\sigma_0} \cdots \partial^{I_q}_{\sigma_q} A_{i_1,\dots,i_q}$$

then the preceding equation (3.30) can be compactly written as $B_0 A_{\sigma_0,\dots,\sigma_q} = 0$. In fact, due to symmetry properties of the type (2.14), namely

$$A^{I_{P(1)},\dots,I_{P(s)}}_{\sigma_{P(1)},\dots,\sigma_{P(s)},i_{Q(1)},\dots,i_{Q(q)}} = (-1)^{|Q|} A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_1,\dots,i_q}, \tag{3.31}$$

we have

$$B_\alpha A_{\sigma_0,\dots,\sigma_q} = 0, \quad \alpha = 0,\dots,q. \tag{3.32}$$

We need a result which is dual to Lemma 2.1:

Lemma 3.3. *The tensor*

$$X \in \mathcal{H}_{s,\underbrace{r,\dots,r}_{k}}$$

satisfies the system of equations

$$B_\alpha X = 0, \quad \alpha = 1,\dots,k \tag{3.33}$$

iff it is of the form

$$X = \prod_{\alpha=1}^{k} B_\alpha X_0 \tag{3.34}$$

where X_0 is arbitrary.

Proof. The implication (3.34) $\Longrightarrow$ (3.33) is trivial because of the relations

$$\{B_\alpha, B_\beta\} = 0. \tag{3.35}$$

We prove the other implication. As a consequence of (3.33) we obtain $B_1 \cdots B_k X = 0$ and Lemma 2.1 can be applied; it follows that X has the generic form

$$X = \sum_{\alpha=1}^{k} B_\alpha X^\alpha$$

for some tensors $X^\alpha \in \mathcal{H}_{s-1,r,\ldots,r,r-1,r,\ldots,r}$ where the entry $r-1$ is on the position α.

One can plug this relation into the initial relation (3.33) and a similar relation is obtained for the tensors X^α. By recurrence, one gets

$$X = \sum_{\alpha_1,\ldots,\alpha_l=1}^{q} B_{\alpha_1} \cdots B_{\alpha_l} X^{\alpha_1,\ldots,\alpha_l} \quad (l = 1,\ldots,k)$$

with $X^{\alpha_1,\ldots,\alpha_l}$ some tensors in $\mathcal{H}_{s-l,r_1,\ldots,r_q}$. In particular, for $l = k$ we obtain that X satisfies (3.34). $\square$

If we apply the lemma above to (3.32), we get $A_{\sigma_0,\ldots,\sigma_q} = 0$ or, explicitly,

$$\partial^{I_0}_{\sigma_0} \cdots \partial^{I_q}_{\sigma_q} A_{i_1,\ldots,i_q} = 0. \tag{3.36}$$

In other words, the functions $A_{i_1,\ldots,i_q}$ are polynomials in y^σ_I ($|I| = r$) of maximal degree q. So, the generic form of these functions is

$$A_{i_1,\ldots,i_q} = \sum_{s=0}^{q} \frac{1}{s!\,(q-s)!} \sum_{|I_1|,\ldots,|I_s|=r} C^{I_1,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s,i_1,\ldots,i_q}\, y^{\sigma_1}_{I_1} \cdots y^{\sigma_s}_{I_s} \tag{3.37}$$

with $C^{I_1,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s,i_1,\ldots,i_q}$ some smooth functions on V^r having symmetry properties of the type (3.31). These functions do not depend on the variables y^σ_I ($|I| = r$) and this justifies the fact that they live on the chart V^{r-1}.

Now we insert this generic expression into the projectability condition (3.29) and we get, in the same way as before,

$$\mathcal{S}^-_{i_0,\ldots,i_q} \mathcal{S}^+_{I_1 p} \delta^p_{i_0} C^{I_1,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s,i_1,\ldots,i_q} = 0, \quad s = 1,\ldots,q-1 \tag{3.38}$$

or, in tensor notation

$$B_1 C_{\sigma_1,\ldots,\sigma_q} = 0, \quad s = 1,\ldots,q-1.$$

In fact, due to the symmetry properties one has from here

$$B_\alpha C_{\sigma_1,\ldots,\sigma_q} = 0, \quad \alpha = 1,\ldots,s; \quad s = 1,\ldots,q-1. \tag{3.39}$$

Using Lemma 3.3 we get from here

$$C_{\sigma_1,\ldots,\sigma_q} = B_1 \cdots B_s \tilde{C}_{\sigma_1,\ldots,\sigma_q}, \quad s = 1,\ldots,q-1.$$

In full index notation this means that we have the generic expression

$$\begin{aligned} C^{J_1 p_1,\dots,J_s p_s}_{\sigma_1,\dots,\sigma_s,i_1,\dots,i_q} &= \mathcal{S}^{+}_{J_1 p_1} \cdots \mathcal{S}^{+}_{J_s p_s} \mathcal{S}^{-}_{i_1,\dots,i_q} \delta^{p_1}_{i_1} \cdots \delta^{p_s}_{i_s} A^{J_1,\dots,J_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_q}, \\ s &= 1,\dots,q \end{aligned} \tag{3.40}$$

where $|J_1| = \cdots = |J_s| = r-1$ and $A^{J_1,\dots,J_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_q}$ are some smooth functions on V^{r-1}, completely antisymmetric in the indices $i_{s+1},\dots,i_q$.

Inserting this expression in (3.37) one immediately obtains

$$A_{i_1,\dots,i_q} = \sum_{s=0}^{q} \frac{1}{s!\,(q-s)!} \sum_{|J_1|,\dots,|J_s|=r-1} \mathcal{S}^{-}_{i_1,\dots,i_q} A^{J_1,\dots,J_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_q} y^{\sigma_1}_{J_1 i_1} \cdots y^{\sigma_s}_{J_s i_s}. \tag{3.41}$$

Let us note that the expression $\mathcal{S}^{-}_{i_1,\dots,i_q} y^{\sigma_1}_{J_1 i_1} \cdots y^{\sigma_s}_{J_s i_s}$ is completely antisymmetric in the couples $(I_1,\sigma_1),\dots,(I_s,\sigma_s)$ so one can consider that the tensors $A^{J_1,\dots,J_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_q}$ have the same property. In the end, it follows that they have the symmetry property (2.14).

The expression for the form ν becomes

$$\begin{aligned} \nu &= \sum_{s=0}^{q} \frac{1}{s!\,(q-s)!} \sum_{|J_1|,\dots,|J_s|=r-1} A^{J_1,\dots,J_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_q} y^{\sigma_1}_{J_1 i_1} \cdots y^{\sigma_s}_{J_s i_s}\, dx^{i_1} \wedge \cdots \wedge dx^{i_q} \\ &= h\tilde{\nu}_V \end{aligned} \tag{3.42}$$

where

$$\begin{aligned} \tilde{\nu}_V \equiv \sum_{s=0}^{q} \frac{1}{s!\,(q-s)!} \sum_{|J_1|,\dots,|J_s|=r-1} & A^{J_1,\dots,J_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_q}\, dy^{\sigma_1}_{J_1} \wedge \cdots \wedge dy^{\sigma_s}_{J_s} \\ & \wedge dx^{i_{s+1}} \wedge \cdots \wedge dx^{i_q}. \end{aligned} \tag{3.43}$$

This finishes the proof if we take into account that the relation (3.29) is equivalent to the projectability condition. □

Now we are ready to obtain the most general form of a variationally trivial Lagrangian. First we note that we have:

Proposition 3.4. *The Lagrange form $\lambda \in \Omega^r_{n,X}$ is variationally trivial iff*

$$E_n([\lambda]) = 0. \tag{3.44}$$

Proof. It is an immediate consequence of the definition of a variationally trivial Lagrange form and of Theorem 2.3. □

The central result now follows:

Theorem 3.5. *Let $\lambda \in \Omega^r_{n,X}$ be a variationally trivial Lagrange form. Then for every point $j^r_x\gamma \in J^rY$ there exists a neighbourhood V of $\gamma(x) \in Y$ and an n-form ρ_V defined in the chart (V^{r-1},ψ^{r-1}) such that we have: (1) $\lambda = h\rho_V$ in the chart (V^r,ψ^r); (2) $d\rho_V = 0$. Conversely, if such a local form ρ_V exists, then the form λ is variationally trivial.*

Proof. (i) According to the previous proposition and applying the exactness of the variational sequence (Theorem 2.2), there exists $\eta \in \Omega^r_{n-1}$ such that

$$[\lambda] = [d\eta],$$

or, equivalently,

$$\lambda - d\eta \in \theta^r_n = \Omega^r_{n(c)}. \tag{3.45}$$

As a consequence we have

$$(\pi^{r+1,r})^*\lambda = h\lambda = hd\eta.$$

In particular, it follows that the n-form $hd\eta$ must be $\pi^{r+1,r}$-projectable. We can apply Proposition 3.1 and rewrite (3.45) as

$$\lambda - d\nu \in \Omega^r_{n(c)} \tag{3.46}$$

for some basic form ν. From here we get

$$h\lambda = hd\nu \tag{3.47}$$

or, using [9, Prop. 3.1],

$$(\pi^{r+1,r})^*\lambda = hd\nu. \tag{3.48}$$

Let us note that that this relation is completely equivalent to the initial condition of variational triviality.

(ii) Next, one sees that from (3.47) it follows in particular that the n-form $d\nu$ is $\pi^{r+1,r}$-projectable. We can apply Proposition 3.2 and obtain that $\nu = h\tilde{\nu}_V$ for some form $\tilde{\nu}_V$ defined on the chart (V^{r-1}, ψ^{r-1}). If we now define

$$\rho_V \equiv d\tilde{\nu}_V \tag{3.49}$$

then we obtain after some computation that

$$(\pi^{r+1,r})^*(\lambda - h\rho_V) = 0$$

and (1) from the statement follows. The definition (3.49) guarantees that we also have (2) from the statement. □

Remark 3.6. A statement of the type appearing in the theorem is, in fact, valid for every $\lambda \in \Omega^r_{q,X}$ $(q \leqslant n)$ such that $E_q([\lambda]) = 0$.

The theorem we just have proved has the following consequence (see also [2, Thm. 4.3]).

Corollary 3.7. *Any variationally trivial Lagrange form $\lambda \in \Omega^r_{n,X}$ can be locally written as a total exterior derivative of a local form $\omega_V \in \mathfrak{J}^r_{n-1}$*

$$\lambda = D\omega_V. \tag{3.50}$$

Proof. In the proof of the preceding theorem we restrict, eventually, the chart V^{r-1} and we have $\rho_V = d\eta_V$ for some $(n-1)$-form on V^{r-1}. Then, according to the definition (2.21) of the total exterior derivative we have the formula from the statement with $\omega_V \equiv h\eta_V \in \mathfrak{J}^r_{n-1}$. □

One can now obtain the most general form of a variationally trivial local Lagrangian.

Theorem 3.8. *Any variationally trivial local Lagrangian of order r has the following form in the chart* (V^r, ψ^r)

$$L = \sum_{s=0}^{n} \frac{1}{s!\,(n-s)!} \sum_{|I_1|,\ldots,|I_s|=r-1} \mathcal{L}^{I_1,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s,i_{s+1},\ldots,i_n} \mathfrak{J}^{\sigma_1,\ldots,\sigma_s,i_{s+1},\ldots,i_n}_{I_1,\ldots,I_s}. \tag{3.51}$$

Here we have defined

$$\mathfrak{J}^{\sigma_1,\ldots,\sigma_s,i_{s+1},\ldots,i_n}_{I_1,\ldots,I_s} \equiv \varepsilon^{i_1,\ldots,i_n} \prod_{l=1}^{s} y^{\sigma_l}_{I_l i_l} \quad (s = 0,\ldots,n) \tag{3.52}$$

and the functions $\mathcal{L}$ *are given by*

$$\begin{aligned} \mathcal{L}^{I_1,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s i_{s+1},\ldots,i_n} \equiv{}& \sum_{k=1}^{s} (-1)^{k-1} \partial^{I_k}_{\sigma_k} A^{I_1,\ldots,\hat{I}_k,\ldots,I_s}_{\sigma_1,\ldots,\hat{\sigma}_k,\ldots,\sigma_s,i_{s+1},\ldots,i_n} \\ &+ \sum_{k=s+1}^{q} (-1)^{k-1} d'_{i_k} A^{I_1,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s,i_{s+1},\ldots,\hat{i}_k,\ldots,i_n}; \end{aligned} \tag{3.53}$$

the expressions $A^{I_1,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s i_{s+1},\ldots,i_{n-1}}$ *are smooth functions on* V^{r-1} *satisfying the symmetry property* (2.14) *and* $d'_j = d^{r-1}_j$ *is the corresponding formal derivative on* V^{r-1}.

Proof. It is convenient to introduce the forms

$$\begin{aligned} \mathcal{A} \equiv \sum_{s=0}^{n-1} \frac{1}{s!\,(n-1-s)!} \sum_{|I_1|,\ldots,|I_s|=r-1} & A^{I_1,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s,i_{s+1},\ldots,i_{n-1}} dy^{\sigma_1}_{I_1} \wedge \cdots \wedge dy^{\sigma_s}_{I_s} \\ & \wedge dx^{i_{s+1}} \wedge \cdots \wedge dx^{i_{n-1}} \end{aligned}$$

and

$$\begin{aligned} \mathcal{F} \equiv \sum_{s=0}^{n} \frac{1}{s!\,(n-s)!} \sum_{|I_1|,\ldots,|I_s|=r-1} & \mathcal{L}^{I_1,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s,i_{s+1},\ldots,i_n} dy^{\sigma_1}_{I_1} \wedge \cdots \wedge dy^{\sigma_s}_{I_s} \\ & \wedge dx^{i_{s+1}} \wedge \cdots \wedge dx^{i_n}. \end{aligned}$$

Then one finds by direct computation that

$$\mathcal{F} = d\mathcal{A} + \text{contact terms}.$$

Next, one takes in Theorem 3.5 $\tilde{\nu}_V = \mathcal{A}$ and it follows that $\rho_V = \mathcal{F} +$ contact terms. Finally, by direct computation one discovers that $\lambda = h\rho_V$ has the expression (3.51). □

Remark 3.9. Let us note that the expressions (3.53) are of the same type as those given by (3.5).

The expressions (3.52) defined above are called *hyper-Jacobians* [4, 11] (see these references for similar results). It is immediate that they have antisymmetry properties of the type (2.14).

Now we give another argument for the converse statement from the preceding theorem. First we have:

Corollary 3.10. *The local expression of a variationally trivial Lagrangian* (3.51) *can be rewritten as*

$$L = d_j V^j \tag{3.54}$$

where V^j are some smooth functions on V^r.

Proof. Using the notation introduced in the proof above let us define the local expressions

$$V^j \equiv \varepsilon^{j,i_1,\dots,i_{n-1}} \sum_{s=0}^{n} \frac{1}{s!\,(n-1-s)!} \sum_{|I_1|,\dots,|I_s|=r-1} A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_{n-1}} y^{\sigma_1}_{I_1 i_1} \cdots y^{\sigma_s}_{I_s i_s}$$

on V^r. One checks now that the formula from the statement is true. □

Now we indeed have:

Theorem 3.11. *The expression* (3.51) *is variationally trivial.*

Proof. We have according to the preceding corollary $E_\sigma(L) = E_\sigma(d_j V^j) = 0$ because we can apply (2.26). □

Remark 3.12. Some globalization of the results above can be found in [16] (see Theorem 5 and Corollary 1 from this reference). In particular, for $r = 1$ one obtains known results [17, 5, 12, 13, 10].

We now present an interesting cohomological fact connected to the equation (3.53). First, we write it in a more compact way. If $A_s \equiv (A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_{n-1}})$ is a system of tensors, then we define new tensors

$$(\delta_1 A)^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_n} \equiv \sum_{k=1}^{s} (-1)^{k-1}\, \partial^{I_k}_{\sigma_k} A^{I_1,\dots,\hat{I}_k,\dots,I_s}_{\sigma_1,\dots,\hat{\sigma}_k,\dots,\sigma_s,i_{s+1},\dots,i_n} \tag{3.55}$$

and

$$(\delta_2 A)^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_n} \equiv \sum_{k=s+1}^{n} (-1)^{k-1}\, d'_{i_k} A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,\hat{i}_k,\dots,i_n}. \tag{3.56}$$

In this way we have introduced two interesting operators $\delta_{1,2}$; we also define

$$\delta \equiv \delta_1 + \delta_2. \tag{3.57}$$

Then we can write the equation (3.53) in a compact way as

$$L_k = \delta A_k. \tag{3.58}$$

As one might suspect, the operators $\delta_{1,2}$ satisfy relations characterizing a cohomological bicomplex,

$$(\delta_1)^2 = 0, \quad (\delta_2)^2 = 0, \quad \delta_1\delta_2 + \delta_2\delta_1 = 0, \tag{3.59}$$

which can be easily proved by direct computations. As a consequence, we have

$$\delta^2 = 0 \tag{3.60}$$

and (3.58) implies

$$\delta \mathcal{L}_k = 0. \tag{3.61}$$

Not very surprisingly this relation is equivalent to (3.58):

Lemma 3.13. *If the system of tensors* $\mathcal{L}_k$ *satisfy the equation* (3.61), *then it is of the form* (3.58).

Proof. We consider the maximal value $k = n + 1$ in (3.61) and we get

$$\begin{aligned} \delta \mathcal{L}_{n+1} = 0 \;&\Leftrightarrow\; \delta_1 \mathcal{L}_{n+1} = 0 \\ &\Leftrightarrow\; \sum_{k=1}^{n+1} (-1)^{k-1}\, \partial^{I_k}_{\sigma_k} \mathcal{L}^{I_1,\dots,\hat{I}_k,\dots,I_{n+1}}_{\sigma_1,\dots,\hat{\sigma}_k,\dots,\sigma_{n+1}} = 0. \end{aligned}$$

The relation above can be "solved" with a usual homotopy operator. If we define, for any set of $n-1$ multi-indices $I_1, \dots, I_{n-1}$, $|I_1| = \cdots = |I_{n-1}| = r - 1$, the expressions

$$\lambda^{I_1,\dots,I_{n-1}}_{\sigma_1,\dots,\sigma_{n-1}} \equiv \int_0^1 dt\, t^{n-1} \sum_{|I_0|=r-1} y^{\sigma_0}_{I_0} \mathcal{L}^{I_0,\dots,I_{n-1}}_{\sigma_0,\dots,\sigma_{n-1}} \circ \chi_r$$

where

$$\chi_r(x^i, y^\sigma_j, \cdots, y^\sigma_{j_1,\dots,j_{r-1}}) = (x^i, y^\sigma_j, \cdots, y^\sigma_{j_1,\dots,j_{r-2}}, t y^\sigma_{j_1,\dots,j_{r-1}}),$$

then we easily get that

$$\mathcal{L}^{I_1,\dots,I_n}_{\sigma_1,\dots,\sigma_n} = (\delta_1 \lambda)^{I_1,\dots,I_n}_{\sigma_1,\dots,\sigma_n} = (\delta \lambda)^{I_1,\dots,I_n}_{\sigma_1,\dots,\sigma_n}.$$

So, we have (3.58) for $k = n$. Now we proceed by induction downwards on the index k. Suppose we have (3.58) for $k = n, n-1, \dots, l+1$; we prove it for $k = l$. For this we take $k = l + 1$ in (3.61). After some rearrangements and use of the formulæ (3.59) we get

$$(\delta_1(\mathcal{L} - \delta_2 \lambda))^{I_1,\dots,I_{l+1}}_{\sigma_1,\dots,\sigma_{l+1},i_{l+2},\dots,i_{n+1}} = 0.$$

But this relation can be analysed with a homotopy operator of the same type as above leading to the relation (3.58) for $k = l$. □

We close this section with some properties of the hyper-Jacobians. We begin with

Proposition 3.14. *The hyper-Jacobians have an antisymmetry properties of the type* (2.14) *and are traceless, i.e., by contracting any of the indices* $i_{s+1}, \dots, i_n$ *in* $\partial^{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_n}_{I_1,\dots,I_s}$ *with any of the indices contained in the multi-indices* $I_1, \dots, I_s$ *one obtains zero.*

Proof. It is elementary by direct computations from the definition. □

So, one knows that the hyper-Jacobians are not linearly independent because of the antisymmetry (2.14) and of the tracelessness condition. We prove here that in fact this is a complete set of linear conditions; more precisely we have:

Theorem 3.15. *Let $\mathcal{L}_k \equiv (\mathcal{L}^{I_1,\dots,I_k}_{\sigma_1,\dots,\sigma_k,i_{k+1},\dots,i_q})$ be functions depending smoothly on the jet bundle extension variables $(x^i, y^\sigma, y^\sigma_j, \dots, y^\sigma_{j_1,\dots,j_{s-1}})$, having the symmetry property* (2.14) *and satisfying the tracelessness condition*

$$B^*_\alpha \mathcal{L}_k = 0. \tag{3.62}$$

Suppose we have

$$\sum_{|I_1|,\dots,|I_k|=s-1} \mathcal{L}^{I_1,\dots,I_k}_{\sigma_1,\dots,\sigma_k,i_{k+1},\dots,i_q} \mathfrak{J}^{\sigma_1,\dots,\sigma_k,i_{k+1},\dots,i_n}_{I_1,\dots,I_k} = 0 \tag{3.63}$$

(with $\mathfrak{J}$ the hyper-Jacobians of order s). Then we have

$$\mathcal{L}^{I_1,\dots,I_k}_{\sigma_1,\dots,\sigma_k,i_{k+1},\dots,i_q} = 0.$$

Proof. We apply the derivative operator $\partial^{I_1}_{\sigma_1}$, $|I_1| = s$, to the equation (3.63) and obtain after some computations

$$\sum_{|I_2|,\dots,|I_k|=s-1} (B_1\mathcal{L})^{I_1,\dots,I_k}_{\sigma_1,\dots,\sigma_k,i_{k+1},\dots,i_{q+1}} \mathfrak{J}^{\sigma_2,\dots,\sigma_k,i_{k+1},\dots,i_{n+1}}_{I_2,\dots,I_k} = 0.$$

Continuing by recurrence we finally get

$$B_1 \cdots B_k \mathcal{L}_k = 0. \tag{3.64}$$

Because the expressions $\mathcal{L}_k$ are traceless we can apply [9, Appendix, Corollary 8.4] and conclude that we have in fact $\mathcal{L}_k = 0$. □

Remark 3.16. If we do not impose the tracelessness condition, then we can use Lemma 2.1 to obtain that the expressions $\mathcal{L}_k$ are sums of δ-terms.

4. The dependence of a variationally trivial Lagrangian on the higest-order derivatives

In this section we analyse the following problem. We have discovered that a variationally trivial Lagrangian depends on the highest order derivatives through some very particular polynomial expressions—the hyper-Jacobians. The problem is to obtain a system of partial differential equations which is compatible only with this structure. We will show in fact that if we isolate from the triviality condition $E_\sigma(\mathcal{L}) = 0$, $\sigma = 1, \dots, m$ those equations involving only the highest-order derivatives, then this system of partial differential equations gives exactly a hyper-Jacobian dependence. This type of result also appears in [1]; as before we obtain here much simpler proofs.

Proposition 4.1. *The local conditions expressing the fact that a certain local Lagrangian $\mathcal{L}$ is variationally trivial*

$$E_\sigma(\mathcal{L}) = 0, \quad \sigma = 1, \ldots, m \tag{4.1}$$

on V^s $(s > 2r)$ are equivalent to

$$\mathcal{S}^+_{p_1,\ldots,p_r,j_r}\, \partial^{p_1,\ldots,p_r}_\rho\, \partial^{j_1,\ldots,j_r}_\sigma L = 0. \tag{4.2}$$

and a system of partial differential equation involving explicitly partial derivatives of order lower that r of $\mathcal{L}$.

Proof. One starts directly from the definition of the Euler–Lagrange expressions

$$\begin{aligned}
E_\sigma(L) &= \sum_{k=0}^{r}(-1)^k\, d^s_{j_1}\cdots d^s_{j_k}\partial^{j_1,\ldots,j_k}_\sigma L \\
&= \sum_{k=0}^{r-1}(-1)^k\, d^s_{j_1}\cdots d^s_{j_k}\partial^{j_1,\ldots,j_k}_\sigma L + (-1)^r\, d^s_{j_1}\cdots d^s_{j_r}\partial^{j_1,\ldots,j_r}_\sigma L \\
&= \sum_{k=0}^{r-1}(-1)^k\, d^s_{j_1}\cdots d^s_{j_k}\partial^{j_1,\ldots,j_k}_\sigma L \\
&\quad + (-1)^r\, d^s_{j_1}\cdots d^s_{j_{r-1}}(d^{r-1}_{j_r} + y^\nu_{j_r,p_1,\ldots,p_r}\partial^{p_1,\ldots,p_r}_\nu)\partial^{j_1,\ldots,j_r}_\sigma L.
\end{aligned}$$

These expressions are first order polynomials in the coordinates y^ρ_J, $|J| = r+1$; the coefficients of the term of first degree can be obtained if we apply a partial derivative of order $r+1$ to the equation above. One notices in this way that the coefficient of $y^\nu_{j_r,p_1,\ldots,p_r}$ is exactly the system (4.2). So we obtain that $\mathcal{L}$ satisfies (4.2) and

$$\sum_{k=0}^{r-1}(-1)^k\, d^s_{j_1}\cdots d^s_{j_k}\partial^{j_1,\ldots,j_k}_\sigma L + (-1)^r\, d^s_{j_1}\cdots d^s_{j_{r-1}} d^{r-1}_{j_r}\partial^{j_1,\ldots,j_r}_\sigma L = 0$$

which certainly contains partial derivatives of order lower than r. □

So, we have to analyse the system (4.2). We start with the following result from [3]. The proof is very simple and it is included for the sake of completeness.

Proposition 4.2. *Let T^j, $j = 1, \ldots, n$, be some smooth local functions on $J^r_n Y$ satisfying the system of equations*

$$S^+_{Ij}\partial^I_\sigma T^j = 0, \quad |I| = r. \tag{4.3}$$

Then T^j are polynomials in y^σ_J, $|J| = r$, of maximal degree $n-1$.

Proof. We define for every vector $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$ the differential operator

$$\partial^X_\sigma \equiv \sum_{|I|=r} X_I \partial^I_\sigma; \quad X_I \equiv \prod_{i\in I} X_i. \tag{4.4}$$

Then the equation from the statement is equivalent to

$$X_j \partial^X_\sigma T^j = 0, \quad \forall X \in \mathbb{R}^n. \tag{4.5}$$

Let us define now the expressions

$$G_{\sigma_1,\ldots,\sigma_n}(X^1,\ldots,X^n,Y) \equiv \partial^{X^1}_{\sigma_1} \cdots \partial^{X^n}_{\sigma_n} T^j Y_j \tag{4.6}$$

where $X^1,\ldots,X^n, Y$ are arbitrary vectors from $\mathbb{R}^n$. If the vectors $X^1,\ldots,X^n$ are linearly independent, then they are a basis in $\mathbb{R}^n$, so one can express Y as a linear combination of them. As a consequence of the equation (4.5) one gets easily that

$$G_{\sigma_1,\ldots,\sigma_n}(X^1,\ldots,X^n,Y) = 0$$

in this case. So, outside the hyperplane determined by the condition that the vectors $X^1,\ldots,X^n$ are linearly *dependent*, the relation above is true. By continuity, this relation is true for all vectors $X^1,\ldots,X^n, Y$. Because these vectors can be arbitrary, this relation is equivalent to

$$\partial^{X^1}_{\sigma_1} \cdots \partial^{X^n}_{\sigma_n} T^j = 0$$

which is the assertion from the statement. □

The analysis of the system (4.3) can be pushed further.

Proposition 4.3. *Let T^j, $j = 1,\ldots,n$, be some smooth local functions on $J^r_n Y$ satisfying the system of equations* (4.3)

$$S^+_{Ij} \partial^I_\sigma T^j = 0, \quad |I| = r.$$

Then they have the generic expression

$$T^{i_n} = \sum_{k=0}^{n-1} \sum_{|I_1|=\cdots=|I_k|=r-1} \mathcal{T}^{I_1,\ldots,I_k}_{\sigma_1,\ldots,\sigma_k,i_{k+1},\ldots,i_{n-1}} \mathcal{J}^{\sigma_1,\ldots,\sigma_k,i_{k+1},\ldots,i_n}_{I_1,\ldots,I_k} \tag{4.7}$$

where $\mathcal{J}$'s are the hyper-Jacobians of order r and the expressions $\mathcal{T}^{I_1,\ldots,I_k}_{\sigma_1,\ldots,\sigma_k,i_{k+1},\ldots,i_{n-1}}$ are smoothly depending on the variables $(x^i, y^\sigma, y^\sigma_j,\ldots,y^\sigma_{j_1,\ldots,j_{r-1}})$ and can be chosen to have symmetry properties of the type (2.14).

Proof. According to the preceding proposition, if the system (4.3) is true, then we must have

$$T^j = \sum_{k=0}^{n-1} \sum_{|I_1|=\cdots=|I_k|=r} A^{\{j\},I_1,\ldots,I_k}_{\sigma_1,\ldots,\sigma_k} y^{\sigma_1}_{I_1} \cdots y^{\sigma_k}_{I_k} \tag{4.8}$$

with the expressions $A^{\{j\},I_1,\ldots,I_k}_{\sigma_1,\ldots,\sigma_k}$ completely symmetric in the couples $(I_1,\sigma_1),\ldots,(I_k,\sigma_k)$ and depending smoothly on the variables $(x^i, y^\sigma, y^\sigma_j,\ldots,y^\sigma_{j_1,\ldots,j_{r-1}})$.

We plug the expression (4.8) into the original equation (4.3) and easily obtain the following conditions on the function coefficients:

$$\mathcal{S}^+_{I_1 j} A^{\{j\},I_1,\ldots,I_k}_{\sigma_1,\ldots,\sigma_k} = 0, \quad k = 1,\ldots,n-1. \tag{4.9}$$

This equation can be written in a compact form using, as many times before, the creation and annihilation operators

$$C_1 A_{\sigma_1,\dots,\sigma_k} = 0 \tag{4.10}$$

where we have defined

$$C_\alpha \equiv b_i^{*(\alpha)}\, a^{(0)i}, \qquad \alpha = 1,\dots,k. \tag{4.11}$$

Here the fermionic index is given by the first entry j and the bosonic indices are given by the families of multi-indices $I_1,\dots,I_k$. If we work with dual tensors

$$\tilde{A}^{I_1,\dots,I_k}_{\sigma_1,\dots,\sigma_k,i_2,\dots,i_n} \equiv \varepsilon_{i_1,\dots,i_n}\, A^{\{i_1\},I_1,\dots,I_k}_{\sigma_1,\dots,\sigma_k} \tag{4.12}$$

then we can use Lemma 6.1 to rewrite the preceding equation as follows

$$B_1 \tilde{A}_{\sigma_1,\dots,\sigma_k} = 0 \tag{4.13}$$

where the operators B_α are the familiar ones defined by (2.15). Moreover, because of the symmetry properties of the tensors A and $\tilde{A}$ we have in fact

$$B_\alpha \tilde{A}_{\sigma_1,\dots,\sigma_k} = 0, \qquad \alpha = 1,\dots,k. \tag{4.14}$$

If we use now Lemma 3.3 we get from here that

$$\tilde{A} = B_1 \cdots B_n \tilde{\mathcal{T}} \Longleftrightarrow A = C_1 \cdots C_1 \mathcal{T} \tag{4.15}$$

with the detailed form

$$\begin{aligned} &A^{\{j_0\},\{I_1'j_1\},\dots,\{I_k'j_k\}}_{\sigma_1,\dots,\sigma_k} = \mathcal{S}^+_{I_1'j_1} \cdots \mathcal{S}^+_{I_k'j_k}\, \tilde{\mathcal{T}}^{\{j_0,\dots,j_k\},I_1',\dots,I_k'}_{\sigma_1,\dots,\sigma_k}, \\ &|I_1'| = \cdots = |I_k'| = r - 1 \end{aligned} \tag{4.16}$$

where the fermionic indices are $j_0,\dots,j_k$ and the bosonic indices are contained in the other multi-indices. This gives

$$T^{j_0} = \sum_{k=0}^{n-1} \sum_{|I_1|=\cdots=|I_k|=r-1} \tilde{\mathcal{T}}^{\{j_0,\dots,j_k\},I_1,\dots,I_k}_{\sigma_1,\dots,\sigma_k}\, y^{\sigma_1}_{I_1j_1} \cdots y^{\sigma_k}_{I_kj_k}. \tag{4.17}$$

If we use the equality

$$\tilde{\mathcal{T}}^{\{j_0,\dots,j_k\},I_1,\dots,I_k}_{\sigma_1,\dots,\sigma_k} \sim \varepsilon^{j_0,\dots,j_{n-1}}\, \mathcal{T}^{I_1,\dots,I_k}_{\sigma_1,\dots,\sigma_k,j_{k+1},\dots,j_{n-1}}$$

then we obtain from the preceding formula (up to some constant factors)

$$T^{j_0} = \sum_{k=0}^{n-1} \sum_{|I_1|=\cdots=|I_k|=r-1} \tilde{\mathcal{T}}^{I_1,\dots,I_k}_{\sigma_1,\dots,\sigma_k,i_{k+1},\dots,i_{n-1}}\, \varepsilon^{j_0,\dots,j_{n-1}}\, y^{\sigma_1}_{I_1j_1} \cdots y^{\sigma_k}_{I_kj_k};$$

using the definition of the hyper-Jacobians this is, up to some signs, exactly the formula to be proved (4.7). □

Remark 4.4. The converse of this proposition is not true in general.

Now we have

Theorem 4.5. *The dependence on the highest-order derivatives of a variationally trivial Lagrangian is given by*

$$L = \sum_{s=0}^{n} \sum_{|I_1|,\ldots,|I_s|=r-1} \mathcal{L}^{I_1,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s,i_{s+1},\ldots,i_n} \mathfrak{J}^{\sigma_1,\ldots,\sigma_s,i_{s+1},\ldots,i_n}_{I_1,\ldots,I_s} \tag{4.18}$$

where $\mathfrak{J}^{\sigma_1,\ldots,\sigma_s,i_{s+1},\ldots,i_n}_{I_1,\ldots,I_s}$ are the hyper-Jacobians of order r and $\mathcal{L}$'s are smooth functions on V^{r-1} satisfying the symmetry property (2.14).

Proof. We inspect the system (4.2) giving the dependence on the highest-order derivatives of a variationally trivial Lagrangian and note that the proposition above can be applied with $T^j \to \partial^{jI}_{\sigma} L$ for any fixed multi-index I of length $r-1$. So, according to (4.7) we obtain that L satisfies

$$\partial^{jI}_{\sigma} L = \sum_{k=0}^{n-1} \sum_{|I_1|=\cdots=|I_k|=r-1} \mathcal{L}^{I,I_1,\ldots,I_k}_{\sigma,\sigma_1,\ldots,\sigma_k,i_{k+1},\ldots,i_{n-1}} \mathfrak{J}^{\sigma_1,\ldots,\sigma_k,i_{k+1},\ldots,i_{n-1},j}_{I_1,\ldots,I_k},$$

where $\mathcal{L}^{I_1,\ldots,I_k}_{\sigma_1,\ldots,\sigma_k,i_{k+1},\ldots,i_n}$ are smoothly depending on the variables $(x^i, y^\sigma, y^\sigma_j, \ldots, y^\sigma_{j_1,\ldots,j_{r-1}})$ and can be chosen to have symmetry properties of the type (2.14).

We apply to this equation the symmetrization operator $\mathcal{S}^+_{jI}$ and get after some computations

$$\partial^{I_0}_{\sigma_0} L = \sum_{k=0}^{n-1} \sum_{|I_1|=\cdots=|I_k|=r-1} (B_0\mathcal{L})^{I_0,\ldots,I_k}_{\sigma_0,\ldots,\sigma_k,i_{k+1},\ldots,i_n} \mathfrak{J}^{\sigma_1,\ldots,\sigma_k,i_{k+1},\ldots,i_n}_{I_1,\ldots,I_k}. \tag{4.19}$$

We must integrate this system of partial differential equations. It is not very hard to prove that this system satisfies Frobenius condition of integrability. So the solution of this system exists and can be found by summing to the general solution of the homogeneous equation $\partial^{I_0}_{\sigma_0} L = 0$ a particular solution of the non-homogeneous equation (4.19). But it is easy to see that the formula (4.18) is such a particular solution. □

Remark 4.6. We have succeeded to find the dependence of locally trivial Lagrangian on the highest-order derivatives without using the exactness of the variational sequence.

5. Locally variational differential equations

The definitions for a general differential equation and for a locally variational differential equation have been given previously (see the formulæ (2.19) and resp. (2.25)). We want to analyse the general structure of a locally variational differential equation along the same lines of argument as in the previous section. It is not possible to obtain the most general expression for such an object (as we have been able to obtain in the case of variationally trivial Lagrangians) but one can produce a generic expression of the same type as (4.18) [1]. There are at least two possibilities to produce this result. One is to mimick, as closely as possible, the proof from Section 3; the proof is instructive but rather long and can be found in the electronic form of [9].

However, there is a much simpler way to obtain the dependence on the highest order derivatives of some arbitrary system of partial differential equations which are locally variational using the results from the preceding section. We will give below this line of argument. We start with the following result.

Proposition 5.1. *Let $T_\sigma, \sigma = 1, \ldots, m$ be the components of an differential equation which is locally differential. Then the following equations are satisfied*

$$\partial^I_\nu T_\sigma - (-1)^s \partial^I_\sigma T_\nu = 0, \quad |I| = s \tag{5.1}$$

and

$$\mathcal{S}^+_{Ij} \partial^J_\rho \partial^{jI}_\nu T_\sigma = 0, \quad |I| = s - 1, \quad |J| = s. \tag{5.2}$$

Moreover, these are the only consequences of the Helmholtz equations (5.3) *which involve only the highest order partial derivatives.*

Proof. We consider Helmholtz–Sonin equations (see (2.27))

$$\partial^J_\nu T_\sigma = (-1)^{|J|} E^J_\sigma(T_\nu), \quad |J| \leqslant s \tag{5.3}$$

taking into account the definition of the Lie–Euler operators (2.22). The equation (5.1) is nothing but the equation (5.3) for $|I| = s$. The equation (5.3) for $|I| = s - 1$ is

$$\partial^I_\nu T_\sigma - (-1)^{s-1}\Big[\partial^I_\sigma T_\nu - s\Big(d^s_j + \sum_{|J|=s} y^\rho_{jJ}\partial^J_\rho\Big)\partial^{jI}_\sigma T_\nu\Big] = 0, \quad |I| = s - 1.$$

The left hand side is a first order polynomial in the coordinates y^ρ_J, $|J| = s+1$; the coefficient of the first degree term can be obtained if we apply a partial derivative of order $s+1$ to the equation above. In this way the second equation from the statement (5.2) is obtained. If we use (5.2) in the preceding equation, we are left again with an equation containing only the expressions y^σ_I, $|I| \leqslant s$.

Now, one must proceed in the same way with the rest of the equations (5.3), i.e., one must extract in the same way from these equations with $|I| = s-2, \ldots, 0$ only equations involving y^σ_I, $|I| \leqslant s$. But it is not very hard to see that in this way there will be no other equation involving only the highest order derivatives. □

Then we have the following theorem:

Theorem 5.2. *The dependence on the highest-order derivatives of the components of a variationally local differential equation of order s is given by*

$$T_\sigma = \sum_{k=0}^{n} \sum_{|I_1|,\ldots,|I_k|=r-1} \mathcal{T}^{I_1,\ldots,I_k}_{\sigma,\nu_1,\ldots,\nu_k,i_{k+1},\ldots,i_n} \mathcal{J}^{\nu_1,\ldots,\nu_k,i_{k+1},\ldots,i_n}_{I_1,\ldots,I_k} \tag{5.4}$$

where $\mathcal{J}^{\nu_1,\ldots,\nu_k,i_{k+1},\ldots,i_n}_{I_1,\ldots,I_k}$ are the hyper-Jacobians of order s and $\mathcal{T}$ are smooth functions on V^{s-1} satisfying the symmetry property (2.14); *this symmetry property leaves aside the index σ.*

Proof. From the relation (5.2), it follows that we can use the Proposition 4.3 with $T^j \to \partial_\nu^{jI} T_\sigma$ with fixed I, ν and σ. We get a formula of the type (4.7)

$$\partial_\nu^{jI} T_\sigma = \sum_{k=0}^{n-1} \sum_{|I_1|,\ldots,|I_k|=r-1} \mathcal{T}^{I,I_1,\ldots,I_k}_{\sigma,\nu,\sigma_1,\ldots,\sigma_k,i_{k+1},\ldots,i_{n-1}} \mathcal{J}^{\sigma_1,\ldots,\sigma_k,i_{k+1},\ldots,i_{n-1},j}_{I_1,\ldots,I_k}.$$

If we apply the operator $\mathcal{S}^+_{jJ}$ we obtain, as in the end of the preceding section, the equations

$$\partial^{I_0}_{\sigma_0} T_\nu = \sum_{k=0}^{n-1} \sum_{|I_1|,\ldots,|I_k|=r-1} (B_0\mathcal{T})^{I_0,\ldots,I_k}_{\nu,\sigma_0,\ldots,\sigma_k,i_{k+1},\ldots,i_n} \mathcal{J}^{\sigma_1,\ldots,\sigma_k,i_{k+1},\ldots,i_n}_{I_1,\ldots,I_k}.$$

This system can be integrated and the formula (5.4) is obtained. □

Remark 5.3. One can see, as in Theorem 4.5, that the expression given by the formula (5.4) satisfies the system (5.2), i.e., (5.4) is the general solution of (5.2).

Remark 5.4. The functions $\mathcal{T}^{I_1,\ldots,I_s}_{\sigma,\nu_1,\ldots,\nu_s,i_{s+1},\ldots,i_n}$ appearing in the statement cannot be completely arbitrary because we did not use all the Helmholtz–Sonin equations; for instance, we did not use (5.1).

Remark 5.5. For the case $r = 2$ analysed in detail in [7] it is possible to use completely the Helmholtz–Sonin equations involving the highest order derivatives. One obtains that T_σ has a expression as in the statement of the theorem, but the coefficients $\mathcal{T}^{l_1,\ldots,l_s}_{\sigma,\nu_1,\ldots,\nu_s,i_{s+1},\ldots,i_n}$ can be chosen to be completely symmetric in the indices $\sigma, \nu_1, \ldots, \nu_s$, completely antisymmetric in the indices $l_1, \ldots, l_s$ and traceless.

6. Appendix

In the case of a fermionic Fock space, one can introduce an important operation, called *Hodge dualization.* We give the definition in coordinate notation although a basis-independent formulation is available. For various notations see [9, Appendix]. It is important that we consider only the finite-dimensional case. It is convenient to make the distinction between the vector space $\mathbb{R}^n$ with vectors of the form $f = (f^i)_{i=1,\ldots,n}$ and the dual space denoted by $\tilde{\mathbb{R}}^n$ with vectors of the form $\tilde{f} = (f_i)_{i=1,\ldots,n}$; a similar convention is made with respect to the corresponding Fock spaces $\mathcal{F}^-(\mathbb{R}^n)$ and $\mathcal{F}^-(\tilde{\mathbb{R}}^n)$. One knows that $\mathbb{R}^n$ and $\tilde{\mathbb{R}}^n$ are (canonically) isomorphic. It is noticeable that the same assertion is true for the corresponding Fock spaces, namely we have

Lemma 6.1. *The transformation $D : \mathcal{F}^-(\mathbb{R}^n) \to \mathcal{F}^-(\tilde{\mathbb{R}}^n)$ given by*

$$(Df)_{i_{k+1},\ldots,i_n} \equiv \frac{1}{(n-k)!}\varepsilon_{i_1,\ldots,i_n} f^{i_1,\ldots,i_k}, \quad k = 0, \ldots, n \tag{6.1}$$

is isomorphism mapping $\mathcal{F}_k$ in $\tilde{\mathcal{F}}_{n-k}$. Its inverse is $D' : \mathcal{F}^-(\tilde{\mathbb{R}}^n) \to \mathcal{F}^-(\mathbb{R}^n)$ given by

$$(D'f)^{i_1,\ldots,i_k} \equiv \frac{1}{k!}\varepsilon^{i_1,\ldots,i_n} f_{i_{k+1},\ldots,i_n}, \quad k = 0, \ldots, n. \tag{6.2}$$

Moreover if we denote the creation and the annihilation operators on the dual Fock space by the tilde sign, then one has

$$Da^i D'|_{\tilde{\mathfrak{F}}_{n-k}} = (-1)^k \tilde{a}^{*i}; \quad Da_i^* D'|_{\tilde{\mathfrak{F}}_{n-k}} = (-1)^k \tilde{a}^i; \quad i = 1, \ldots, n. \tag{6.3}$$

In other words, the Hodge operation transforms the annihilation operators into creation operators and *vice versa*. The proof is elementarily done by direct computations. We only mention that the usual notation is $D = *$.

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